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A parabolic variational inequality arising from the valuation of strike reset options ☆

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Abstract

A strike reset option is an option that allows its holder to reset the strike price to the prevailing underlying asset price at a moment chosen by the holder. The pricing model of the option can be formulated as a one-dimensional parabolic variational inequality, or equivalently, a free boundary problem, where the free boundary just corresponds to the optimal reset strategy adopted by the holder of the option. This paper is concerned with the theoretical analysis of the model. The existence and uniqueness of the solution are established. Furthermore, we study properties of the free boundary. The monotonicity and C^∞ smoothness of the free boundary are proven in some situations.

MSC: 35R35

Keywords: Option pricing; Strike reset option; Free boundary; Variational inequality; American option

1. Introduction

A well-known parabolic variational inequality arising from financial markets is the valuation model of the American option. In this paper, we consider a similar model which is derived from the valuation of another option, called the strike reset option. The option allows its holder

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to reset the strike price to the prevailing underlying asset price at the moment of resetting (see [2,5–7,10,18,20,21]). Subject to specified provisions of the option contract, the moment to reset can be either (i) at some pre-determined dates, or (ii) chosen optimally by the holder. The option pricing in the case (i) is relatively easy because its governing equation is a linear PDE (see [5]). This paper is focused on the case (ii) for which a natural problem is how to optimally determine the reset moment, in addition to pricing the option. We shall see later on that the option pricing problem in this situation leads to a one-dimensional parabolic variational inequality, or equivalently, a free boundary problem, where the free boundary just corresponds to the optimal reset strategy.

Let us briefly introduce the modelling of the option pricing. Without loss of generality, we assume the initial strike price $X = 1$. Let T be the expiration date and t be the calendar time. Let $V(S,\tau)$ denote the option value, where S and $\tau = T - t$ are the underlying price and the time to expiry, respectively. We always confine our discussion within the Black–Scholes framework where the risk-neutral price process of the underlying is assumed to follow a geometric Brownian motion:

$$\frac{dS_t}{S_t} = \bar{r}\,dt + \sigma\,dZ_t.$$

Here Z_t is the standard Wiener process, $\sigma > 0$ is the constant volatility, and $\bar{r} = r - q$ is the difference of the constant riskless interest rate $r > 0$ and the constant dividend yield $q \geqslant 0$. To establish the pricing model, a critical observation is that at the reset moment, the option becomes an at-the-money put option whose price amounts to $SP(\tau)$ (see [12] or [19]), where

$$P(\tau) = e^{-r\tau} N\left(-\frac{\bar{r} - \sigma^2/2}{\sigma}\sqrt{\tau}\right) - e^{-q\tau} N\left(-\frac{\bar{r} + \sigma^2/2}{\sigma}\sqrt{\tau}\right), \tag{1.1}$$

with

$$N(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\zeta} e^{-\eta^2/2}\,d\eta.$$

Because the reset moment is chosen voluntarily by the holder, this leads to the following optimal stopping problem for the option pricing:

$$V(S,\tau) = \sup_{t^*} \widehat{E}\left[e^{-r(t^*-t)} G\left(S_{t^*}, T - t^*\right) \,\middle|\, S_t = S\right], \quad \tau = T - t,$$

where $\widehat{E}$ is the risk neutral expectation, t^* is the optimal stopping time between t and T, and

$$G(S,\tau) = \begin{cases} SP(\tau), & \text{if } \tau > 0, \\ (1-S)^+, & \text{if } \tau = 0. \end{cases}$$

The option value $V(S,\tau)$ is also the viscosity solution of the variational inequality problem (see [7,20] or [21]):

$$\begin{cases} \partial_\tau V - \frac{\sigma^2}{2} S^2 \partial_{SS} V - \bar{r} S \partial_S V + rV \geqslant 0, & S > 0,\ 0 < \tau \leqslant T, \\ V(S,\tau) \geqslant SP(\tau), & S > 0,\ 0 < \tau \leqslant T, \\ \left[\partial_\tau V - \frac{\sigma^2}{2} S^2 \partial_{SS} V - \bar{r} S \partial_S V + rV\right][V - SP(\tau)] = 0, \\ V(S,0) = (1-S)^+. \end{cases} \tag{1.2}$$

It can be observed that, apart from the obstacle function $SP(\tau)$, this model resembles the well-known American options pricing model.

Most previous work on the model (1.2) is devoted to efficient numerical approaches (see [6, 18,20,21]). One exception is [7] in which the authors develop an analytical framework to analyze properties of the optimal reset strategy (i.e., the free boundary). A particular interesting result obtained in the paper is that the optimal reset strategy sensitively depends on the sign of $\bar{r}$.

This paper is concerned with the theoretical analysis to the model (1.2). We aim to establish the uniqueness and existence of $W_p^{2,1}$ solution, and to exploit more properties of the free boundary. Since the difference between the model (1.2) and the American option pricing model lies only in obstacle functions, it appears that the former would not cause more difficulties than the latter which has been widely studied by numerous researchers (see [3,4,15,19], and references therein). However, the seemingly slight difference indeed results in a rather complicated analysis with regard to the model (1.2). Later we will see that this is primarily because the temporal derivative of the obstacle function $SP(\tau)$ has a singularity at $\tau = 0$.

The rest of this paper is organized as follows. In Section 2, we prove the existence and uniqueness of the solution. We shall mainly deal with the singularity mentioned above, as well as the nonsmooth initial value and the unbounded solution domain. Section 3 is addressed to the C^∞ smoothness and the monotonicity of the free boundary. It contains two subsections. In Section 3.1 we discuss the case $|\bar{r}| \leqslant \sigma^2/2$, where the key step is to achieve (3.15). In Section 3.2 we consider the case of $\bar{r} < -\sigma^2/2$, in which the free boundary is shown to be bounded and C^∞ smooth for all time, although we cannot achieve the global monotonicity of the free boundary.

2. The existence and uniqueness of $W_p^{2,1}$ solution

Apart from the singularity of obstacle function, the model (1.2) has another two features often appearing in option pricing problems: (i) the solution domain is unbounded; (ii) the initial value function is only Lipschitz continuous. First, let us remove the singularity of initial value. To do that, we take into account the function

$$U(S,\tau) = V(S,\tau) - V_0(S,\tau),$$

where $V_0(S,\tau)$ is the price function of the European vanilla put option satisfying the Black–Scholes equation (see [19]):

$$\begin{cases} \partial_\tau V_0 - \frac{\sigma^2}{2} S^2 \partial_{SS} V_0 - \bar{r} S \partial_S V_0 + rV_0 = 0, & S > 0,\ 0 < \tau \leqslant T, \\ V_0(S,0) = (1-S)^+. \end{cases} \tag{2.1}$$

Then $U(S,t)$ is governed by the following parabolic variational inequality with zero initial value condition:

$$\begin{cases} \partial_\tau U - \frac{\sigma^2}{2} S^2 \partial_{SS} U - \bar{r} S \partial_S U + rU \geqslant 0, & S > 0,\ 0 < \tau \leqslant T, \\ U(S,\tau) \geqslant SP(\tau) - V_0(S,\tau), & S > 0,\ 0 < \tau \leqslant T, \\ \big[\partial_\tau U - \frac{\sigma^2}{2} S^2 \partial_{SS} U - \bar{r} S \partial_S U + rU\big]\big[U - SP(\tau) + V_0(S,\tau)\big] = 0, \\ U(S,0) = 0. \end{cases} \tag{2.2}$$

For later reference, we point out that the celebrated Black–Scholes formula gives an explicit expression of $V_0(S,\tau)$ as follows (see [12] or [19]):

$$V_0(S,\tau) = e^{-r\tau} N\left(-\frac{\ln S + (\bar{r} - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right) - Se^{-q\tau} N\left(-\frac{\ln S + (\bar{r} + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right). \tag{2.3}$$

Note that

$$P(\tau) = V_0(1,\tau). \tag{2.4}$$

By the transformation

$$\begin{aligned} &x = \ln S, \qquad v_0(x,\tau) = e^{q\tau - x} V_0\big(e^x,\tau\big), \\ &u(x,\tau) = e^{q\tau - x} U\big(e^x,\tau\big), \qquad Q(\tau) = e^{q\tau} P(\tau), \end{aligned} \tag{2.5}$$

(2.1) and (2.2) can be reduced to

$$\begin{cases} \mathcal{L} v_0 = 0, & x \in \mathbb{R}^1,\ 0 < \tau \leqslant T, \\ v_0(x,0) = (e^{-x} - 1)^+ \end{cases} \quad \text{and} \tag{2.6}$$

$$\begin{cases} \mathcal{L} u \geqslant 0, \qquad u - u_0 \geqslant 0, & x \in \mathbb{R}^1,\ 0 < \tau \leqslant T, \\ (\mathcal{L} u)(u - u_0) = 0, & x \in \mathbb{R}^1,\ 0 < \tau \leqslant T, \\ u(x,0) = 0, \end{cases} \tag{2.7}$$

respectively, where $\mathcal{L} u = \partial_\tau u - \frac{\sigma^2}{2} \partial_{xx} u - (\bar{r} + \frac{\sigma^2}{2}) \partial_x u$ and

$$u_0(x,\tau) = Q(\tau) - v_0(x,\tau).$$

We will prove that the problem (2.7) has a unique solution in the function class

$$\mathcal{A} = W_p^{2,1}\big(\Omega_T^R\big) \cap W_{q,\mathrm{loc}}^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T) \cap L^\infty(\Omega_T),$$

where $1 < p < 2$, $2 < q < +\infty$, $\Omega_T = \mathbb{R}^1 \times (0,T)$ and $\Omega_T^R = (-R,R) \times (0,T)$ for any $R > 0$.

We now focus on the new obstacle function $u_0(x,\tau)$. From (2.3)–(2.5), we have

$$v_0(x,\tau) = e^{-x - \bar{r}\tau} N\left(-\frac{x + (\bar{r} - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right) - N\left(-\frac{x + (\bar{r} + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right), \quad \text{and} \tag{2.8}$$

$$v_0(0,\tau) = Q(\tau). \tag{2.9}$$

It is easy to check by (2.8)

$$\partial_x v_0(x,\tau) \leqslant 0, \quad x \in \mathbb{R}^1,\ 0 < \tau \leqslant T. \tag{2.10}$$

In fact (2.10) can also be deduced by the comparison principle for the PDE model (2.6). Therefore, it follows

$$\partial_x u_0(x,\tau) = -\partial_x v_0(x,\tau) \geqslant 0,$$

which combines with $u_0(0,\tau) = Q(\tau) - v_0(0,\tau) = 0$ to yield

$$\begin{cases} u_0(x,\tau) > 0, & \text{if } x > 0, \\ u_0(x,\tau) = 0, & \text{if } x = 0, \\ u_0(x,\tau) < 0, & \text{if } x < 0, \end{cases} \tag{2.11}$$

since $u \in W^{2,1}_{q,\mathrm{loc}}(\Omega_T) \cap C(\overline{\Omega}_T) \cap L^\infty(\Omega_T)$, applying the Alexandrof–Bakel'man–Pucci (ABP) minimum principle (see [17]) to the model (2.7) leads to

$$u \geqslant 0. \tag{2.12}$$

Combining (2.11) with (2.12), we are able to replace the obstacle function u_0 by

$$u_0^+ = \max(u_0, 0) = \begin{cases} u_0, & x > 0,\ 0 < \tau \leqslant T, \\ 0, & x < 0,\ 0 < \tau \leqslant T. \end{cases}$$

In order to analyze the behavior of the free boundary conveniently we introduce the problem

$$\begin{cases} \mathcal{L}u \geqslant 0, \qquad u - u_0^+ \geqslant 0, & x \in \mathbb{R}^1,\ 0 < \tau \leqslant T, \\ \mathcal{L}u(u - u_0^+) = 0, & x \in \mathbb{R}^1,\ 0 < \tau \leqslant T, \\ u(x,0) = 0. \end{cases} \tag{2.13}$$

It is clear that a solution to (2.7) also solves (2.13). On other hand, we will see in Theorem 2.4 that problem (2.13) has a unique solution in the function class $W^{2,1}_{q,\mathrm{loc}}(\Omega_T) \cap C(\overline{\Omega}_T) \cap L^\infty(\Omega_T)$, and in a similar way the problem (2.7) can be shown to have a unique solution in the same function class. Hence the solution to (2.13) solves (2.7) as well. In another words, problem (2.7) and problem (2.13) are equivalent. In the following, we only need to focus on problem (2.13).

In order to prove the existence of the solution, we consider the following penalty approximation of problem (2.13)

$$\begin{cases} \mathcal{L}u_\varepsilon + \beta_\varepsilon(u_\varepsilon - u_0^+) = 0, & x \in \mathbb{R}^1,\ 0 < \tau \leqslant T, \\ u_\varepsilon(x,0) = 0, \end{cases} \tag{2.14}$$

where the penalty function $\beta_\epsilon(t)$ is given by Fig. 1, satisfying

$$\begin{aligned} &0 < \varepsilon < 2, \qquad \beta_\epsilon(t) \in C^2(-\infty,+\infty), \\ &\beta_\epsilon(t) \leqslant 0, \qquad \beta_\epsilon(0) = -1, \\ &0 \leqslant \beta'_\varepsilon(t) \leqslant 2/\varepsilon, \qquad \beta''_\epsilon \leqslant 0, \end{aligned}$$

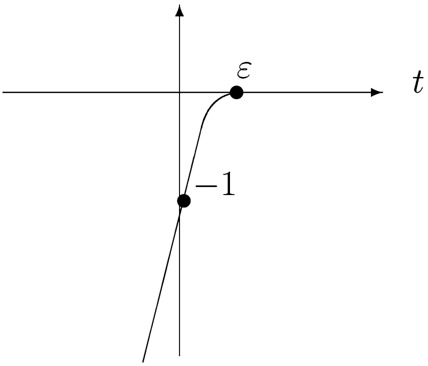

Fig. 1.

and

$$\beta_\varepsilon(t)=\begin{cases}0, & t\geqslant\varepsilon,\\ (2/\varepsilon-1)t-1, & t\leqslant\varepsilon/2.\end{cases}\tag{2.15}$$

Since the solution domain of problem (2.14) is unbounded, we instead take into account the problem in a bounded domain, i.e.,

$$\begin{cases}\mathcal{L}u_{\varepsilon,R}+\beta_\varepsilon(u_{\varepsilon,R}-u_0^+)=0, & (x,\tau)\in\Omega_T^R,\\ u_{\varepsilon,R}(x,0)=0, & -R<x<R,\\ u_{\varepsilon,R}(R,\tau)=u_0(R,\tau), & \\ u_{\varepsilon,R}(-R,\tau)=0, & \end{cases}\tag{2.16}$$

where $\Omega_T^R=(-R,R)\times(0,T]$, $R>0$.

Lemma 2.1. *For any fixed* ε, R, *problem* (2.16) *has a unique solution*

$$u=u_{\varepsilon,R}\in W_p^{2,1}\bigl(\Omega_T^R\bigr)\cap C^{2+\alpha,1+\alpha/2}\bigl(\Omega_T^R\bigr)\cap C\bigl(\overline{\Omega_T^R}\bigr),\quad 1<p<2,\ 0<\alpha<1/2.$$

Proof. We will employ the Schauder fixed point theorem (see [11, Chapter 11]) to prove the existence of the solution to the nonlinear problem (2.16).

Set $B=C(\overline{\Omega}_T^R)$ and $\mathcal{D}=\{w\in B\mid w\geqslant 0\}$. Note that $\mathcal{D}$ is a closed and convex set in B. We now define an operator as follows: for any $w\in\mathcal{D}$ given, let $u=\mathcal{F}w$ be the solution to the following linear PDE problem:

$$\begin{cases}\mathcal{L}u=-\beta_\varepsilon(w-u_0^+), & \text{in } \Omega_T^R,\\ u(x,0)=0, & -R<x<R,\\ u(R,\tau)=u_0(R,\tau), & \\ u(-R,\tau)=0. & \end{cases}\tag{2.17}$$

Due to (2.8) and (2.9), we know that

$$Q(\tau)=e^{-\bar{r}\tau}N(-d_2)-N(-d_1),$$

where

$$d_1 = \frac{\bar{r} + \sigma^2/2}{\sigma}\sqrt{\tau}, \qquad d_2 = \frac{\bar{r} - \sigma^2/2}{\sigma}\sqrt{\tau}.$$

It can be checked that

$$Q'(\tau) = O\left(\tau^{-1/2}\right) \quad \text{as } \tau \to 0^+, \qquad Q(\tau) = O\left(\tau^{1/2}\right) \quad \text{as } \tau \to 0^+.$$

Clearly $v_0(R,\tau) \in C^\infty[0,T]$. We then have $u_0(R,\tau) = Q(\tau) - v_0(R,\tau) \in W_p^1(0,T)$ $(1 < p < 2)$. Note that $u_0^+ \in C^{\alpha,\alpha/2}(\overline{\Omega}_T^R)$. Hence problem (2.17) has a unique solution $u \in W_p^{2,1}(\Omega_T^R)$ (see [16]), and

$$\begin{aligned} |u|_{W_p^{2,1}(\Omega_T^R)} &\leqslant C\left(\left|\beta_\varepsilon\left(w - u_0^+\right)\right|_{L^\infty(\Omega_T^R)} + \left|u_0(R,\tau)\right|_{W_p^1(0,T)}\right) \\ &\leqslant C\left(\left|\beta_\varepsilon\left(-u_0^+\right)\right|_{L^\infty(\Omega_T^R)} + \left|u_0(R,\tau)\right|_{W_p^1(0,T)}\right), \end{aligned} \tag{2.18}$$

where C depends on ε, R and $1 < p < 2$.

Next, we are going to show that the operator $\mathcal{F}$ has the following three properties when $3/2 < p < 2$:

(1) $\mathcal{F}(\mathcal{D}) \subset \mathcal{D}$;
(2) $\mathcal{F}(\mathcal{D})$ is precompact in B;
(3) $\mathcal{F}$ is continuous.

In fact: (1) $\beta_\varepsilon \leqslant 0$ and $u_0(R,\tau) \geqslant 0$ imply $\mathcal{L}u \geqslant 0$ and $u \geqslant 0$, respectively. Then we obtain $\mathcal{F}(\mathcal{D}) \subset \mathcal{D}$ because any $W_p^{2,1}$ function is continuous when $p > 3/2$.

(2) Due to (2.18), there exists a constant $C > 0$, such that

$$|u|_{W_p^{2,1}(\Omega_T^R)} \leqslant C.$$

Thanks to the imbedding theorem, $\mathcal{F}(\mathcal{D})$ is precompact in B when $3/2 < p < 2$.

(3) To obtain the continuity of $\mathcal{F}$, it suffices to show that if

$$w_j \longrightarrow w, \quad w_j, w \in \mathcal{D}, \quad \text{and}$$

$$\mathcal{F}(w_j) = u_j, \qquad \mathcal{F}(w) = u,$$

then

$$\lim_{j\to\infty} u_j = u \quad \text{in } \mathcal{D}.$$

Indeed, due to (2.17), $u_j - u$ satisfies

$$\begin{cases} \mathcal{L}(u_j - u) = -\beta_\varepsilon'(\cdot)(w_j - w) & \text{in } \Omega_T^R, \\ (u_j - u) = 0 & \text{on } \partial\Omega_T^R. \end{cases}$$

So,

$$|u_j - u|_{L^\infty(\Omega_T^R)} \leqslant C|w_j - w|_{L^\infty(\Omega_T^R)},$$

which implies the continuity of $\mathcal{F}$.

Now we can use the Schauder fixed point theorem and the imbedding theorem to infer that problem (2.16) has a solution $u \in W_p^{2,1}(\Omega_T^R) \cap C^{\alpha,\alpha/2}(\overline{\Omega_T^R})$ when $1 < p < 2$, $0 < \alpha < 1/2$. The further $C^{2+\alpha,1+\alpha/2}(\Omega_T^R)$ regularity comes from the $C^{2+\alpha,1+\alpha/2}$ interior estimation, and the uniqueness follows by the monotonicity of β_ε. $\square$

Lemma 2.2. *There exist constants $M_1, M_2 > 0$, such that*

$$0 \leqslant u_{\varepsilon,R} \leqslant M_1\sqrt{\tau} + \varepsilon \qquad in\ \Omega_T^R, \tag{2.19}$$

$$-M_2 \leqslant \sqrt{\tau}\,\beta_\varepsilon\big(u_{\varepsilon,R} - u_0^+\big) \leqslant 0 \quad in\ \Omega_T^R, \tag{2.20}$$

where M_1, M_2 are independent of ε, R.

Moreover, if $\bar r < 0$, then

$$0 \leqslant u_{\varepsilon,R} \leqslant e^{-\bar r\tau} + \varepsilon \quad in\ \Omega_T^R. \tag{2.21}$$

Proof. The minimum principle for the model (2.16) implies $u_{\varepsilon,R} \geqslant 0$. Denote

$$U_t = \big\{(x,\tau) \in \Omega_t^R \bigm| u_{\varepsilon,R}(x,\tau) - u_0^+(x,\tau) > \varepsilon\big\}$$

for any $t > 0$ fixed. Since $\mathcal{L}u_{\varepsilon,R} = 0$ in U_t and $u_{\varepsilon,R} = u_0^+ + \varepsilon$ on the boundary of U_t in Ω_t^R, the boundary and initial value of Ω_t^R and the maximum principle imply

$$u_{\varepsilon,R} \leqslant \big|u_0^+\big|_{C(\overline{\Omega_t^R})} + \varepsilon = \big|(Q - v_0)^+\big|_{C(\overline{\Omega_t^R})} + \varepsilon \leqslant |Q|_{C[0,t]} + \varepsilon \leqslant M_1\sqrt{t} + \varepsilon.$$

Moreover, we have

$$\begin{aligned}\big|Q(\tau)\big|_{C[0,t]} &= \big|e^{q\tau}P(\tau)\big|_{C[0,t]} = \big|e^{-\bar r\tau}N(-d_2) - N(-d_1)\big|_{C[0,t]}\\ &\leqslant \big|e^{-\bar r\tau}\big|_{C[0,t]} \leqslant e^{-\bar r t} \quad \text{if } \bar r < 0.\end{aligned}$$

Thus (2.21) is achieved. Next we prove (2.20). Since $\beta_\varepsilon \leqslant 0$, it suffices to estimate the minimum of β_ε. If $u_{\varepsilon,R} - u_0^+ \geqslant 0$, then we immediately have $\sqrt{\tau}\,\beta_\varepsilon(u_{\varepsilon,R} - u_0^+) \geqslant -\sqrt{T}$ because of the definition of β_ε. And if $u_{\varepsilon,R} - u_0^+$ has a negative minimum, then both $\sqrt{\tau}(u_{\varepsilon,R} - u_0^+)$ and $\sqrt{\tau}\,\beta_\varepsilon(u_{\varepsilon,R} - u_0^+)$ must have a negative minimum as well. Suppose $(x_\varepsilon, \tau_\varepsilon)$ and $(x'_\varepsilon, \tau'_\varepsilon)$ are minimum points of $\sqrt{\tau}(u_{\varepsilon,R} - u_0^+)$ and $\sqrt{\tau}\,\beta_\varepsilon(u_{\varepsilon,R} - u_0^+)$, respectively. According to the boundary conditions in (2.16) we know that $(x_\varepsilon, \tau_\varepsilon)$ must be an interior point of Ω_T^R with $x_\varepsilon > 0$.

Applying the equation in (2.16), we get

$$\begin{aligned}\mathcal{L}\big(\sqrt{\tau}\big(u_{\varepsilon,R}-u_0^+\big)\big)&=\sqrt{\tau}\,\mathcal{L}\big(u_{\varepsilon,R}-u_0^+\big)+\frac{u_{\varepsilon,R}-u_0^+}{2\sqrt{\tau}}\\&=\sqrt{\tau}\,\mathcal{L}u_{\varepsilon,R}-\sqrt{\tau}\,\mathcal{L}\big(Q(\tau)-v_0(x,\tau)\big)+\frac{u_{\varepsilon,R}-u_0^+}{2\sqrt{\tau}}\\&=-\sqrt{\tau}\,\beta_\varepsilon\big(u_{\varepsilon,R}-u_0^+\big)-\sqrt{\tau}\,Q'(\tau)+\frac{u_{\varepsilon,R}-u_0^+}{2\sqrt{\tau}}\end{aligned}\tag{2.22}$$

if $x>0$. Combination of $\mathcal{L}(\sqrt{\tau}(u_{\varepsilon,R}-u_0^+))\leqslant 0$ at $(x_\varepsilon,\tau_\varepsilon)$ and (2.22) yields

$$\begin{aligned}&\sqrt{\tau_\varepsilon}\beta_\varepsilon\big(u_{\varepsilon,R}(x_\varepsilon,\tau_\varepsilon)-u_0^+(x_\varepsilon,\tau_\varepsilon)\big)\\&\quad\geqslant-\sqrt{\tau_\varepsilon}Q'(\tau_\varepsilon)+\frac{u_{\varepsilon,R}(x_\varepsilon,\tau_\varepsilon)-u_0^+(x_\varepsilon,\tau_\varepsilon)}{2\sqrt{\tau_\varepsilon}}\\&\quad=-\sqrt{\tau_\varepsilon}Q'(\tau_\varepsilon)+\frac{u_{\varepsilon,R}(x_\varepsilon,\tau_\varepsilon)-Q(\tau_\varepsilon)+v_0(x_\varepsilon,\tau_\varepsilon)}{2\sqrt{\tau_\varepsilon}}\geqslant-\sqrt{\tau_\varepsilon}Q'(\tau_\varepsilon)-\frac{Q(\tau_\varepsilon)}{2\sqrt{\tau_\varepsilon}}.\end{aligned}\tag{2.23}$$

Notice that when $u_{\varepsilon,R}-u_0^+<0$, it follows from (2.15)

$$\beta_\varepsilon\big(u_{\varepsilon,R}-u_0^+\big)=\left(\frac{2}{\varepsilon}-1\right)\big(u_{\varepsilon,R}-u_0^+\big)-1.\tag{2.24}$$

As a result, for any $(x,\tau)\in\Omega_T^R$,

$$\begin{aligned}&\sqrt{\tau}\,\beta_\varepsilon\big(u_{\varepsilon,R}(x,\tau)-u_0^+(x,\tau)\big)\\&\quad\geqslant\sqrt{\tau_\varepsilon'}\beta_\varepsilon\big(u_{\varepsilon,R}\big(x_\varepsilon',\tau_\varepsilon'\big)-u_0^+\big(x_\varepsilon',\tau_\varepsilon'\big)\big)\\&\quad=\beta_\varepsilon\big[\sqrt{\tau_\varepsilon'}\big(u_{\varepsilon,R}\big(x_\varepsilon',\tau_\varepsilon'\big)-u_0^+\big(x_\varepsilon',\tau_\varepsilon'\big)\big)\big]+\big(1-\sqrt{\tau_\varepsilon'}\big)\qquad\big(\text{by (2.24)}\big)\\&\quad\geqslant\beta_\varepsilon\big[\sqrt{\tau_\varepsilon}\big(u_{\varepsilon,R}(x_\varepsilon,\tau_\varepsilon)-u_0^+(x_\varepsilon,\tau_\varepsilon)\big)\big]+\big(1-\sqrt{\tau_\varepsilon'}\big)\qquad(\text{by monotonicity of }\beta_\varepsilon)\\&\quad=\sqrt{\tau_\varepsilon}\,\beta_\varepsilon\big(u_{\varepsilon,R}(x_\varepsilon,\tau_\varepsilon)-u_0^+(x_\varepsilon,\tau_\varepsilon)\big)+\big(\sqrt{\tau_\varepsilon}-\sqrt{\tau_\varepsilon'}\big)\qquad\big(\text{by (2.24)}\big)\\&\quad\geqslant-\sqrt{\tau_\varepsilon}Q'(\tau_\varepsilon)-\frac{Q(\tau_\varepsilon)}{2\sqrt{\tau_\varepsilon}}+\big(\sqrt{\tau_\varepsilon}-\sqrt{\tau_\varepsilon'}\big)\qquad\big(\text{by (2.23)}\big)\\&\quad\geqslant-M_2,\end{aligned}$$

where the last inequality is due to $Q(\tau)\in C^\infty(0,T]$ and $Q(\tau)=O(\tau^{1/2})$, $Q'(\tau)=O(\tau^{-1/2})$ as $\tau\to0^+$. $\square$

Lemma 2.3. *For any fixed ε, problem* (2.14) *has a unique solution $u_\varepsilon\in W_p^{2,1}(\Omega_T^R)\cap C(\overline{\Omega}_T)$, for $R>0$, $1<p<2$. Moreover,*

$$0 \leqslant u_\varepsilon \leqslant M_1\sqrt{\tau} + \varepsilon \quad \text{in } \Omega_T, \tag{2.25}$$

$$-M_2 \leqslant \sqrt{\tau}\,\beta_\varepsilon\big(u_\varepsilon - u_0^+\big) \leqslant 0 \quad \text{in } \Omega_T. \tag{2.26}$$

where M_1, M_2 are independent of ε.

And if $\bar{r} < 0$, then

$$0 \leqslant u_\varepsilon \leqslant e^{-\bar{r}\tau} + \varepsilon \quad \text{in } \Omega_T^R. \tag{2.27}$$

Proof. From estimates (2.19), (2.20) we know that $u_{\varepsilon,R}$, $\beta_\varepsilon(u_{\varepsilon,R} - u_0^+) \in L^p$, $1 < p < 2$. By virtue of the standard interior estimates and extracting diagonal subsequences, we are able to obtain the existence of the solution. The uniqueness comes from the monotonicity of β_ε, and (2.25)–(2.27) are consequences of (2.19)–(2.21). The details are omitted. □

Theorem 2.4. *Problem* (2.13) *has a unique solution $u \in W^{2,1}_{q,\mathrm{loc}}(\Omega_T) \cap C(\overline{\Omega}_T) \cap L^\infty(\Omega_T)$, $2 \leqslant q < +\infty$, and for any $R > 0$, $u \in W_p^{2,1}(\Omega_T^R)$, $1 < p < 2$. Moreover,*

$$0 \leqslant u(x,\tau) \leqslant M_1\sqrt{\tau}, \quad (x,\tau) \in \Omega_T. \tag{2.28}$$

And if $\bar{r} < 0$, then

$$0 \leqslant u \leqslant e^{-\bar{r}\tau} \quad \text{in } \Omega_T. \tag{2.29}$$

Remark. For the solution of problem (1.2), $V(S,\tau) = e^{-q\tau}Su(\ln S,\tau) + V_0(S,\tau)$, so $V(S,\tau) \in W^{2,1}_{q,\mathrm{loc}}((0,+\infty) \times (0,T]) \cap C(\overline{(0,+\infty) \times (0,T)})$, $2 \leqslant q < +\infty$, and for any $\delta, R > 0$ $u \in W_p^{2,1}((\delta,R) \times (0,T))$, $1 < p < 2$. Moreover,

$$V_0(S,\tau) \leqslant V(S,\tau) \leqslant Se^{-q\tau}M_1\sqrt{\tau} + V_0(S,\tau), \quad (S,\tau) \in (0,+\infty) \times (0,T),$$

where $V_0(S,\tau)$ is defined by (2.3).

Proof of Theorem 2.4. (2.26) implies

$$\big|\beta_\varepsilon\big(u_\varepsilon - u_0^+\big)\big|_{L^p(\Omega_T^R)} \leqslant C$$

for any $1 < p < 2$, $R > 0$. It can be reduced, by (2.14) and (2.25), to

$$|u_\varepsilon|_{W_p^{2,1}(\Omega_T^R)} \leqslant C,$$

where C is independent of ε, but depends on R. Therefore, using the same method as in the proof of Lemma 2.3, we conclude that there exists a subsequence of $\{u_\varepsilon\}$, still denoted by itself for convenience, and $u \in W_p^{2,1}(\Omega_T^R)$, such that:

$$u_\varepsilon \longrightarrow u \quad \text{in } W_p^{2,1}\big(\Omega_T^R\big) \text{ weakly}, \qquad u_\varepsilon \longrightarrow u \quad \text{in } C\big(\overline{\Omega_T^R}\big).$$

Letting $\varepsilon \to 0$ in $\mathcal{L}u_\varepsilon \geqslant 0$, one gets

$$\mathcal{L}u \geqslant 0 \quad \text{in } \Omega_T^R.$$

Since R is arbitrary, we then have

$$\mathcal{L}u \geqslant 0 \quad \text{in } \Omega_T.$$

From (2.26) and the definition of β_ε, we deduce that for any fixed $(x,\tau) \in \Omega_T$ and any $\delta > 0$, $u_\varepsilon(x,\tau) - u_0^+(x,\tau) > -\delta$ when ε is small enough. We then take $\delta \to 0^+$ to get

$$u(x,\tau) \geqslant u_0^+(x,\tau).$$

Next, we prove

$$\mathcal{L}u = 0 \quad \text{in } \{u > u_0^+\}.$$

In fact, for any $(x_0,\tau_0) \in \{u(x,\tau) > u_0^+(x,\tau)\}$, we have

$$u(x_0,\tau_0) > u_0^+(x_0,\tau_0).$$

As a consequence, there exists a $\delta > 0$ such that

$$u_\varepsilon(x_0,\tau_0) > u_0^+(x_0,\tau_0) + \delta$$

for sufficient small ε. So, as $\varepsilon \to 0^+$, we have

$$\beta_\varepsilon\big(u_\varepsilon(x_0,\tau_0) - u_0^+(x_0,\tau_0)\big) \geqslant \beta_\varepsilon(\delta) \longrightarrow 0,$$

which yields the desired result $\mathcal{L}u(x_0,\tau_0) = 0$.

Now we prove $u \in W^{2,1}_{q,\mathrm{loc}}(\Omega_T)$. In fact, from (2.26) we know that $\beta_\epsilon(u_\epsilon - u_0^+)$ is locally bounded on Ω_T, going back to problem (2.14) we have that

$$|u_\epsilon|_{W_q^{2,1}((-R,R)\times(\sigma,T))} \leqslant C,$$

where $2 \leqslant q < +\infty$, $\sigma > 0$ and C is independent of ϵ, therefore $u \in W^{2,1}_{q,\mathrm{loc}}(\Omega_T)$.

Let us move on to the proof of the uniqueness of the solution in the function class $W^{2,1}_{q,\mathrm{loc}}(\Omega_T) \cap C(\overline{\Omega}_T) \cap L^\infty(\Omega_T)$, $2 \leqslant q < +\infty$. Assume that u_1, u_2 are two solutions to problem (2.13). Without loss of generalization, we suppose $\{u_1 > u_2\}$ is not empty. Then

$$u_1 > u_2 \geqslant u_0^+ \quad \text{on } \{u_1 > u_2\}.$$

Hence

$$\mathcal{L}u_1 = 0 \quad \text{in } \{u_1 > u_2\}, \qquad \mathcal{L}u_2 \geqslant 0 \quad \text{in } \{u_1 > u_2\} \quad \text{and}$$

$$\mathcal{L}(u_1 - u_2) \leqslant 0 \quad \text{in } \{u_1 > u_2\}.$$

Applying the ABP maximum principle (see [17]) again, we deduce $u_1 - u_2 \leqslant 0$ in $\{u_1 > u_2\}$, which results in a contradiction.

Finally, we prove (2.28) and (2.29). Letting $\varepsilon \to 0$ in (2.25) and (2.27) we deduce $0 \leqslant u \leqslant M_1\sqrt{\tau}$ and $0 \leqslant u \leqslant e^{-\bar{r}\tau}$, which are desired. □

3. The properties of free boundary

This section is devoted to some analytical properties of the free boundary. We only need to take into consideration the model (2.13). To begin with, we summarize some known results obtained in [7] as follows:

Proposition 3.1. *Let $h(\tau)$ be the free boundary of the model* (2.13) *which is defined by* (3.20). *Then*

(1) $h(0^+) = 0$;
(2) *The behavior of $h(\tau)$ depends on the sign of $\bar{r}$: If $\bar{r} \leqslant 0$, then $h(\tau)$ is finite for all $\tau \in (0, \infty)$ and*

$$\lim_{\tau \to \infty} h(\tau) = \ln\left(1 - \frac{\sigma^2}{2\bar{r}}\right).$$

Especially, $\lim_{\tau\to\infty} h(\tau) = \infty$ *when $\bar{r} = 0$. If $\bar{r} > 0$, then $h(\tau)$ exists only for $\tau \in (0, \tau^*)$, where τ^* is the unique root of the equation*

$$Q'(\tau) = 0 \quad \textit{in } (0, \infty).$$

In the following, we aim to study the monotonicity and smoothness of the free boundary. First, let us consider the case of $|r| \leqslant \sigma^2/2$.

3.1. *The properties of free boundary in the case of $|r| \leqslant \sigma^2/2$*

By the transformation

$$\bar{u}(x,\tau) = e^{\bar{r}\tau} u(x,\tau), \qquad \bar{u}_0(x,\tau) = e^{\bar{r}\tau} u_0(x,\tau) = e^{\bar{r}\tau}\big[Q(\tau) - v_0(x,\tau)\big],$$

we obtain, from (2.13),

$$\begin{cases} \mathcal{L}\bar{u} - \bar{r}\bar{u} \geqslant 0, \qquad \bar{u} - \bar{u}_0^+ \geqslant 0, & x \in \mathbb{R}^1,\ 0 < \tau \leqslant T, \\ (\mathcal{L}\bar{u} - \bar{r}\bar{u})(\bar{u} - \bar{u}_0^+) = 0, & x \in \mathbb{R}^1,\ 0 < \tau \leqslant T, \\ \bar{u}(x,0) = 0. \end{cases} \tag{3.1}$$

Lemma 3.2. *If $\bar{r} \leqslant \sigma^2/2$, then*

$$\partial_\tau \bar{u}_0(x,\tau) > 0, \quad x > 0. \tag{3.2}$$

Proof. Due to (2.8),

$$\begin{aligned}\partial_x \bar{u}_0(x,\tau) &= -\partial_x\big[e^{\bar{r}\tau} v_0(x,\tau)\big]\\ &= -\partial_x\bigg[e^{-x} N\bigg(-\frac{x+(\bar{r}-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\bigg) - e^{\bar{r}\tau} N\bigg(-\frac{x+(\bar{r}+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\bigg)\bigg]\\ &= e^{-x}N(-\bar{d}_2) + e^{-x}\frac{n(\bar{d}_2)}{\sigma\sqrt{\tau}} - e^{\bar{r}\tau}\frac{n(\bar{d}_1)}{\sigma\sqrt{\tau}} = e^{-x}N(-\bar{d}_2), \end{aligned} \tag{3.3}$$

where

$$\bar{d}_1 = \frac{x+(\bar{r}+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \qquad \bar{d}_2 = \frac{x+(\bar{r}-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \qquad n(\eta) = \frac{1}{\sqrt{2\pi}}e^{-\eta^2/2}$$

and the identity $e^{-x}n(\bar{d}_2) = e^{\bar{r}\tau}n(\bar{d}_1)$ is used. Moreover, from (3.3),

$$\partial_{x\tau}\bar{u}_0(x,\tau) = e^{-x}n(\bar{d}_2)\frac{1}{2\sigma\sqrt{\tau}}\bigg(\frac{x}{\tau} - \big(\bar{r}-\sigma^2/2\big)\bigg).$$

If $\bar{r} \leqslant \sigma^2/2$, then $\partial_{x\tau}\bar{u}_0 > 0$ while $x > 0$. This gives the desire result, combining with $\partial_\tau \bar{u}_0(0,\tau) = 0$. □

Lemma 3.3. *If $\bar{r} \leqslant \sigma^2/2$, then*

$$\partial_\tau \bar{u}(0,\tau) \geqslant 0, \quad 0 < \tau \leqslant T. \tag{3.4}$$

Proof. For any $\delta > 0$, the system (3.1) can be rewritten as

$$\begin{cases} (\mathcal{L}-\bar{r})\bar{u}(x,\tau+\delta) \geqslant 0, \qquad \bar{u}(x,\tau+\delta) - \bar{u}_0^+(x,\tau+\delta) \geqslant 0, & x \in \mathbb{R}^1,\ 0<\tau\leqslant T-\delta,\\ [(\mathcal{L}-\bar{r})\bar{u}(x,\tau+\delta)][\bar{u}(x,\tau+\delta) - \bar{u}_0^+(x,\tau+\delta)] = 0, & x\in\mathbb{R}^1,\ 0<\tau\leqslant T-\delta,\\ \bar{u}(x,\delta) \geqslant 0 = \bar{u}(x,0). \end{cases} \tag{3.5}$$

From (3.2) we know that

$$\bar{u}_0^+(x,\tau+\delta) \geqslant \bar{u}_0^+(x,\tau).$$

Applying comparison principle of variational inequalities (3.1) and (3.5) with respect to obstacles and initial values (see [9, Problem 5, p. 80]), we get $\bar{u}(x,\tau+\delta) \geqslant \bar{u}(x,\tau)$ and thus $\partial_\tau\bar{u}(x,\tau) \geqslant 0$. Especially the conclusion holds. □

In what follows we are going to show that if $|\bar{r}| \leqslant \sigma^2/2$, then the free boundary is a smooth and strictly monotonically increasing curve. In the first place, we prove some properties about $Q(\tau)$:

Lemma 3.4.

(1) *There exists a $\tau^{**} > 0$, such that*

$$\frac{d}{d\tau}\big[e^{\bar{r}\tau} Q'(\tau)\big] < 0, \quad 0 < \tau < \tau^{**}; \tag{3.6}$$

moreover, if $|\bar{r}| \leqslant \sigma^2/2$, then $\tau^{**} = +\infty$, i.e.,

$$\frac{d}{d\tau}\big[e^{\bar{r}\tau} Q'(\tau)\big] < 0, \quad 0 < \tau < +\infty. \tag{3.7}$$

(2) *If $\bar{r} > 0$, then there is a constant $\tau^* > 0$, such that*

$$\begin{cases} Q'(\tau) > 0, & 0 < \tau < \tau^*, \\ Q'(\tau) = 0, & \tau = \tau^*, \\ Q'(\tau) < 0, & \tau > \tau^*, \end{cases} \tag{3.8}$$

where τ^* is the same as that given in Proposition 3.1.

(3) *If $\bar{r} = 0$, then*

$$Q'(\tau) > 0, \quad 0 < \tau < +\infty. \tag{3.9}$$

(4) *If $\bar{r} < 0$, then*

$$Q'(\tau) \geqslant \delta_0 e^{-\bar{r}\tau}, \quad 0 < \tau < +\infty, \tag{3.10}$$

where δ_0 is a positive constant.

Proof. The proof of parts (2) and (3) can be found in [7]. We only need to show parts (1) and (4).

(1) Since

$$Q(\tau) = e^{-\bar{r}\tau} N(-d_2) - N(-d_1), \tag{3.11}$$

it is not hard to verify

$$\begin{aligned} Q'(\tau) &= -\bar{r}e^{-\bar{r}\tau} N(-d_2) + e^{-\bar{r}\tau} n(d_2)\left(-\frac{\bar{r} - \sigma^2/2}{\sigma}\right)\frac{1}{2\sqrt{\tau}} - n(d_1)\left(-\frac{\bar{r} + \sigma^2/2}{\sigma}\right)\frac{1}{2\sqrt{\tau}} \\ &= e^{-\bar{r}\tau}\left[-\bar{r}N(-d_2) + \frac{\sigma}{2\sqrt{\tau}} n(d_2)\right], \end{aligned} \tag{3.12}$$

where the identity $n(d_1) = e^{-\bar{r}\tau} n(d_2)$ is used. So,

$$e^{\bar{r}\tau} Q'(\tau) = -\bar{r}N(-d_2) + \frac{\sigma}{2\sqrt{\tau}} n(d_2) \tag{3.13}$$

and

$$\frac{d}{d\tau}\big[e^{\bar{r}\tau}Q'(\tau)\big] = -\bar{r}n(d_2)\left(-\frac{\bar{r}-\sigma^2/2}{\sigma}\right)\frac{1}{2\sqrt{\tau}} - \frac{\sigma}{2\sqrt{\tau}}n(d_2)\frac{(\bar{r}-\sigma^2/2)^2}{2\sigma^2} - \frac{\sigma}{4\tau\sqrt{\tau}}n(d_2)$$
$$= \left(\frac{\bar{r}^2-\sigma^4/4}{4\sigma} - \frac{\sigma}{4\tau}\right)\frac{n(d_2)}{\sqrt{\tau}},$$

which yields the desired result.

(4) If $\bar{r} < 0$, then from (3.14) we have

$$Q'(\tau) \geqslant -\bar{r}e^{-\bar{r}\tau}N(-d_2) \geqslant -e^{-\bar{r}\tau}\frac{\bar{r}}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-\eta^2/2}\,d\eta. \qquad \square$$

Applying (2.11) and (2.12) we know that

$$u(x,\tau) > u_0(x,\tau), \quad x < 0,\ 0 \leqslant \tau \leqslant T.$$

It means that there is no free boundary in the region $\{x < 0\}$, thus we can confine problem (3.1) in the domain $\{x \geqslant 0,\ 0 \leqslant \tau \leqslant T\}$ to analyze the behavior of free boundary. Denote

$$w(x,\tau) = \bar{u}(x,\tau) - \bar{u}_0(x,\tau), \quad x \geqslant 0,\ 0 \leqslant \tau \leqslant T.$$

Then (3.1) becomes

$$\begin{cases} \mathcal{L}w(x,\tau) - \bar{r}w(x,\tau) \geqslant -e^{\bar{r}\tau}Q'(\tau), \quad w(x,\tau) \geqslant 0,\ x > 0,\ 0 < \tau \leqslant T, \\ [\mathcal{L}w(x,\tau) - \bar{r}w(x,\tau) + e^{\bar{r}\tau}Q'(\tau)]w(x,\tau) = 0, \quad x > 0,\ 0 < \tau \leqslant T, \\ w(0,\tau) = \bar{u}(0,\tau), \quad 0 < \tau \leqslant T, \\ w(x,0) = 0, \quad x \geqslant 0. \end{cases} \tag{3.14}$$

Lemma 3.5. *If $|\bar{r}| \leqslant \sigma^2/2$, then the solution to problem* (3.14) *has the following properties*:

$$\partial_\tau w(x,\tau) \geqslant 0, \quad x \geqslant 0,\ 0 \leqslant \tau \leqslant T, \tag{3.15}$$
$$\partial_x w(x,\tau) \leqslant 0, \quad x \geqslant 0,\ 0 \leqslant \tau \leqslant T. \tag{3.16}$$

Proof. For any small $\delta > 0$, $w(x,\tau+\delta)$ satisfies, by (3.14),

$$\begin{cases} \mathcal{L}w(x,\tau+\delta) - \bar{r}w(x,\tau+\delta) \geqslant -e^{\bar{r}(\tau+\delta)}Q'(\tau+\delta), \quad w(x,\tau+\delta) \geqslant 0, \\ \quad x > 0,\ 0 < \tau \leqslant T-\delta, \\ [\mathcal{L}w(x,\tau+\delta) - \bar{r}w(x,\tau+\delta) + e^{\bar{r}(\tau+\delta)}Q'(\tau+\delta)]w(x,\tau+\delta) = 0, \\ \quad x > 0,\ 0 < \tau \leqslant T-\delta, \\ w(0,\tau+\delta) = \bar{u}(0,\tau+\delta), \quad 0 < \tau \leqslant T-\delta, \\ w(x,\delta) \geqslant 0 = w(x,0), \quad x \geqslant 0. \end{cases} \tag{3.17}$$

From (3.4) and (3.7) we have

$$w(0,\tau+\delta) \geqslant w(0,\tau), \quad 0<\tau \leqslant T-\delta,$$
$$-e^{\bar{r}(\tau+\delta)} Q'(\tau+\delta) \geqslant -e^{\bar{r}(\tau)} Q'(\tau), \quad 0<\tau \leqslant T-\delta.$$

Applying comparison principle of variational inequality (see [9, Problem 5, p. 80]) to problems (3.14) and (3.17), we obtain

$$w(x,\tau+\delta) \geqslant w(x,\tau), \quad x \geqslant 0,\ 0 \leqslant \tau \leqslant T-\delta.$$

So (3.15) is obtained. Now we prove (3.16). Note that (2.14) can be rewritten as

$$\begin{cases} \mathcal{L}(u_\varepsilon - u_0) + \beta_\varepsilon(u_\varepsilon - u_0^+) = -Q'(\tau), & (x,\tau) \in \Omega_T, \\ u_\varepsilon(x,0) - u_0(x,0) = (e^{-x}-1)^+. \end{cases} \tag{3.18}$$

Differentiating (3.18) with respect to x, we get

$$\begin{cases} \mathcal{L}[\partial_x(u_\varepsilon - u_0)] + \beta'_\varepsilon(\cdot)\partial_x(u_\varepsilon - u_0) = \beta'_\varepsilon(\cdot)(\partial_x u_0^+ - \partial_x u_0), & (x,\tau) \in \Omega_T, \\ \partial_x u_\varepsilon(x,0) - \partial_x u_0(x,0) \leqslant 0, & x \in \mathbb{R}^1. \end{cases} \tag{3.19}$$

Notice that

$$\beta'_\varepsilon(\cdot)\big(\partial_x u_0^+ - \partial_x u_0\big) = \begin{cases} 0, & x \geqslant 0. \\ \beta'_\varepsilon(\cdot)\partial_x v_0 \leqslant 0, & x < 0. \end{cases}$$

It follows from the maximum principle that

$$\partial_x\big[u_\varepsilon(x,\tau) - u_0(x,\tau)\big] \leqslant 0, \quad x \in \mathbb{R}^1,\ 0<\tau \leqslant T,$$

which gives (3.16). $\square$

(3.16) implies that we may define the free boundary:

$$h(\tau) = \inf\big\{x \mid w(x,\tau) = 0\big\}, \quad 0<\tau \leqslant T. \tag{3.20}$$

From (3.15) and (3.16), we immediately get the following theorem.

Theorem 3.6. *If $0<\bar{r} \leqslant \sigma^2/2$, the free boundary $x = h(\tau)$ is a monotonic increasing curve with respect to τ, $0<\tau<\tau^*$.*

We further exploit more refined properties of $h(t)$.

Theorem 3.7. *If $0<\bar{r} \leqslant \sigma^2/2$, then*

(1) *$h(\tau)$ is continuous in $(0,\tau^*)$;*
(2) *$x = h(\tau)$ has no vertical part, i.e., $h(\tau)$ is strictly increasing;*
(3) *$h(\tau) \in C[0,\tau^*) \cap C^\infty(0,\tau^*)$.*

Proof. (1) If $h(\tau)$ has a discontinuous point τ_0, $0<\tau_0<\tau^*$, then $\lim_{\tau\to\tau_0^-} = x_1 < \lim_{\tau\to\tau_0^+} = x_2$. Due to (3.14), there is a $\delta>0$, such that

$$\begin{cases} \mathcal{L}w - \bar{r}w = -e^{\bar{r}\tau}Q'(\tau), & x_1<x<x_2,\ \tau_0<\tau<\tau_0+\delta, \\ w(x,\tau_0)=0, & x_1<x<x_2. \end{cases} \tag{3.21}$$

Letting $\tau\to\tau_0^+$ in the equation in (3.21) we have from (3.8)

$$\partial_\tau w(x,\tau_0) = -e^{\bar{r}\tau_0}Q'(\tau_0)<0, \quad x_1<x<x_2,$$

which contradicts (3.15).

(2) Suppose that $x=h(\tau)$ has a vertical part, for example, $\{x=x_0,\ \tau_1<\tau<\tau_2\}\subset\{x=h(\tau)\}$, where $\tau_1>0$, $\tau_2<\tau^*$. Then there is a $\delta>0$ such that

$$\begin{cases} \mathcal{L}w - \bar{r}w = -e^{\bar{r}\tau}Q'(\tau), & x_0-\delta<x<x_0,\ \tau_1<\tau<\tau_2, \\ \partial_x w(x_0,\tau)=0, & \tau_1<\tau<\tau_2. \end{cases} \tag{3.22}$$

Differentiating system (3.22) with respect to τ we obtain

$$\begin{cases} \mathcal{L}(\partial_\tau w) - \bar{r}\partial_\tau w = -\frac{d}{d\tau}[e^{\bar{r}\tau}Q'(\tau)]>0, \quad \big(\text{by (3.7)}\big), \\ \quad x_0-\delta<x<x_0,\ \tau_1<\tau<\tau_2, \\ \partial_x(\partial_\tau w)(x_0,\tau)=0, \quad \tau_1<\tau<\tau_2. \end{cases} \tag{3.23}$$

Note that under the condition (3.15) $\partial_\tau w$ is continuous in the domain $\{x\geqslant 0,\ 0<\tau\leqslant T\}$ (see [1, Theorem 1.2]). Hence, $\partial_\tau w$ has a minimum value equal to zero on the interval $\{x=x_0,\ \tau_1<\tau<\tau_2\}$. This is a contradiction with the boundary condition of (3.23).

(3) Clearly $h(\tau)\in C[0,\tau^*)$. The proof for the $C^\infty(0,\tau^*)$ smoothness of $h(\tau)$ is the same as in [8], where the condition (3.15) is crucial. We omit the details. □

Using the same arguments as in the proof of Theorems 3.6 and 3.7 we obtain:

Theorem 3.8. *If $-\sigma^2/2\leqslant\bar{r}\leqslant 0$, then for any $T>0$, $h(\tau)$ is strictly increasing and $h(0)=0$, $h(\tau)\in C[0,T]\cap C^\infty(0,T]$.*

3.2. The properties of free boundary in the case of $\bar{r}<-\sigma^2/2$

We now study the properties of the free boundary in the case of $\bar{r}<-\sigma^2/2$.

Theorem 3.9. *The free boundary $x=h(\tau)$ is bounded, i.e., there is a constant $R_0>0$ independent of T, such that*

$$0<h(\tau)\leqslant R_0, \quad 0<\tau\leqslant T. \tag{3.24}$$

Proof. We construct an auxiliary function of problem (3.14)

$$W(x)=\begin{cases}\dfrac{2k\delta_0}{\sigma^2}\big[k(e^{(R_0-x)/k}-1)+(x-R_0)\big], & 0\leqslant x\leqslant R_0,\\ 0, & x\geqslant R_0.\end{cases}$$

Taking $k=\sigma^2/(2\bar r+\sigma^2)$ and δ_0 to be the same as in (3.10), it is not hard to check that

$$\text{while } 0\leqslant x\leqslant R_0:\quad \partial_x W=\frac{2k\delta_0}{\sigma^2}\big[1-e^{(R_0-x)/k}\big]\leqslant 0,\qquad \partial_{xx}W=\frac{2\delta_0}{\sigma^2}e^{(R_0-x)/k},$$

$$\mathcal{L}W-\bar rW\geqslant -\frac{\sigma^2}{2}\partial_{xx}W-\left(\bar r+\frac{\sigma^2}{2}\right)\partial_x W=-\delta_0\geqslant -e^{\bar r\tau}Q'(\tau),$$

where the last inequality is due to (3.10). From (2.29) we know that $\bar u(0,\tau)=e^{\bar r\tau}u(0,\tau)\leqslant 1$ on $[0,+\infty)$. So we can take R_0 to be sufficiently large such that

$$W(0)\geqslant \sup_{0\leqslant\tau<+\infty}\bar u(0,\tau)=\sup_{0\leqslant\tau<+\infty}w(0,\tau).$$

The comparison principle of variational inequality to problem (3.14) implies $w(x,\tau)\leqslant W(x)$ and thus $h(\tau)\leqslant R_0$. □

Lemma 3.10. *If $\bar r<-\sigma^2/2$, the solution to problem* (3.14) *has the following properties*

$$\partial_\tau w(x,\tau)\geqslant 0,\quad x\geqslant 0,\ 0\leqslant\tau\leqslant\tau^{**},\tag{3.25}$$

$$\partial_x w(x,\tau)\leqslant 0,\quad x\geqslant 0,\ 0\leqslant\tau\leqslant T.\tag{3.26}$$

Proof. (3.25) can be obtained by Lemma 3.3, (3.6) and the same argument as in the proof of (3.15). As for (3.26), the proof is the same as that of (3.16). □

Theorem 3.11. *If $\bar r<-\sigma^2/2$, then $h(\tau)$ is strictly increasing on $[0,\tau^{**}]$, and $h(\tau)\in C[0,T]\cap C^\infty(0,T]$ for any $T>0$.*

Proof. It is easy to see, by (3.25) and (3.26), $h(\tau)$ is strictly increasing on $[0,\tau^{**}]$, $h(\tau)\in C[0,\tau^{**}]\cap C^\infty(0,\tau^{**}]$. Return to problem (2.13) and denote

$$Z(x,\tau)=u(x,\tau)-u_0(x,\tau),\quad x\geqslant 0,\ \tau\geqslant 0.$$

We can rewrite problem (2.13) as a free boundary problem on the interval $[0,\tau^{**}]$

$$\mathcal{L}Z(x,\tau)=-Q'(\tau),\quad 0<x<h(\tau),\ 0<\tau\leqslant\tau^{**},\tag{3.27}$$

$$Z(0,\tau)=u(0,\tau),\quad 0<\tau\leqslant\tau^{**},\tag{3.28}$$

$$Z\big(h(\tau),\tau\big)=0,\quad 0<\tau\leqslant\tau^{*},\tag{3.29}$$

$$\partial_x Z\big(h(\tau),\tau\big)=0,\quad 0<\tau\leqslant\tau^{**}.\tag{3.30}$$

Differentiating (3.30) with respect to τ leads to

$$\partial_{xx} Z\big(h(\tau),\tau\big)h'(\tau)+\partial_{x\tau} Z\big(h(\tau),\tau\big)=0. \tag{3.31}$$

Letting $x=h(\tau)$ in Eq. (3.27) we have

$$\partial_{xx} Z\big(h(\tau),\tau\big)=\frac{2}{\sigma^2}Q'(\tau).$$

Plugging it into (3.31) we obtain

$$h'(\tau)=-\frac{\sigma^2}{2}\frac{\partial_{x\tau} Z}{Q'(\tau)}.$$

Denote $\theta(x,\tau)=\partial_\tau Z(x,\tau)$. Differentiating system (3.27)–(3.30) with respect to τ, we get a free boundary problem of Stefan type for $(\theta(x,\tau),h(\tau))$:

$$\mathcal{L}\theta(x,\tau)=-Q''(\tau),\quad 0<x<h(\tau),\ 0<\tau\leqslant\tau^{**}, \tag{3.32}$$

$$\theta(0,\tau)=\partial_\tau u(0,\tau),\quad 0<\tau\leqslant\tau^{**}, \tag{3.33}$$

$$\theta\big(h(\tau),\tau\big)=0,\quad 0<\tau\leqslant\tau^{**}, \tag{3.34}$$

$$h'(\tau)=-\frac{\sigma^2}{2}\frac{\partial_x\theta}{Q'(\tau)},\quad 0<\tau\leqslant\tau^{**}, \tag{3.35}$$

$$h(0)=0. \tag{3.36}$$

Since there is no singularity for both $Q'(\tau)$ and $Q''(\tau)$ on the interval $[\tau^{**},T]$ and $Q'(\tau)$ has a positive lower bound δ_0 by (3.10), we are able to find a smooth solution $(\theta(x,\tau),g(\tau))$ for the one-phase Stefan problem (see [13,14])

$$\mathcal{L}\theta(x,\tau)=-Q''(\tau),\quad 0<x<g(\tau),\ \tau^{**}-\delta<\tau\leqslant T, \tag{3.37}$$

$$\theta(0,\tau)=\partial_\tau u(0,\tau),\quad \tau^{**}-\delta<\tau\leqslant T, \tag{3.38}$$

$$\theta\big(g(\tau),\tau\big)=0,\quad \tau^{**}-\delta<\tau\leqslant T, \tag{3.39}$$

$$g'(\tau)=-\frac{\sigma^2}{2}\frac{\partial_x\theta}{Q'(\tau)},\quad \tau^{**}-\delta<\tau\leqslant T, \tag{3.40}$$

$$\theta(x,\tau^{**}-\delta)=\partial_\tau Z(x,\tau^{**}-\delta),\quad 0\leqslant x\leqslant h(\tau^{**}-\delta), \tag{3.41}$$

$$g(\tau^{**}-\delta)=h(\tau^{**}-\delta), \tag{3.42}$$

where $\delta\leqslant\tau^{**}/2$ is a small positive number. Note that boundary value in (3.38) is

$$\partial_\tau u(0,\tau)=\partial_\tau\big[e^{-\bar r\tau}\bar u(0,\tau)\big]=e^{-\bar r\tau}\big[\partial_\tau\bar u(0,\tau)-\bar r\bar u(0,\tau)\big]>0,\quad \tau^{**}-\delta\leqslant\tau\leqslant T,$$

and the initial value in (3.41) is

$$\partial_\tau Z(x,\tau)=\partial_\tau\big[e^{-\bar r\tau}w(x,\tau)\big]=e^{-\bar r\tau}\big[\partial_\tau w(x,\tau)-\bar r w(x,\tau)\big]>0,\quad \tau=\tau^{**}-\delta.$$

Now we attempt to show a priori estimate that there exists a $x_0 > 0$ independent of T, such that

$$h(\tau) \geqslant x_0, \quad \tau^{**} - \delta \leqslant \tau \leqslant T. \tag{3.43}$$

To do that, we construct an auxiliary function of system (3.14) in the domain $\{(x, \tau) \mid x \geqslant 0,\ \tau^{**} - \delta \leqslant \tau \leqslant T\}$:

$$W(x) = \begin{cases} A(x - x_0)^2, & 0 \leqslant x \leqslant x_0, \\ 0, & x > x_0, \end{cases}$$

where $x_0 > 0$, $A > 0$ are to be determined. It is not hard to check that

$$\text{while } 0 \leqslant x \leqslant x_0\text{:} \quad \partial_x W = 2A(x - x_0) \leqslant 0, \qquad \partial_{xx} W = 2A,$$
$$\mathcal{L}W - \bar{r}W \leqslant -\frac{\sigma^2}{2}\partial_{xx} W - \bar{r}W = -A\big[\sigma^2 + \bar{r}(x - x_0)^2\big].$$

By (3.14) we have

$$e^{\bar{r}\tau} Q'(\tau) \leqslant -\bar{r} + \frac{\sigma}{2\sqrt{\tau^{**} - \delta}} \leqslant M_3, \quad \tau^{**} - \delta \leqslant \tau \leqslant T.$$

Take $A = 2M_3/\sigma^2$. It follows that

$$-A\sigma^2/2 = -M_3 \leqslant -e^{\bar{r}\tau} Q'(\tau), \quad \tau^{**} - \delta \leqslant \tau \leqslant T.$$

We then choose a sufficiently small $x_0 < h(\tau^{**} - \delta)/2 = b$ such that

$$Ax_0^2 \leqslant w(b, \tau^{**} - \delta) \quad \text{and} \quad \bar{r}x_0^2 \geqslant -\sigma^2/2.$$

Due to (3.15) and (3.16), we have the following hold while $0 \leqslant x \leqslant x_0$, $\tau^{**} - \delta \leqslant \tau \leqslant T$

$$\mathcal{L}W \leqslant -A\sigma^2/2 \leqslant -e^{\bar{r}\tau} Q'(\tau).$$

$$W(x, \tau) \leqslant Ax_0^2 \leqslant \min_{0 \leqslant x \leqslant b} w(x, \tau^{**} - \delta) \leqslant \min_{\tau^{**} - \delta \leqslant \tau \leqslant T} w(0, \tau).$$

Applying the comparison principle of variational inequality to problem (3.14) in the domain of $\{(x, \tau) \mid 0 < x < x_0,\ \tau^{**} - \delta < \tau \leqslant T\}$, then we have $w(x, \tau) \geqslant W(x)$, which implies $h(\tau) \geqslant x_0$.

Based on estimate (3.43) we deduce that problem (3.37)–(3.42) has a global smooth solution (see [13,14]). Due to the uniqueness of variational inequality, we have

$$h(\tau) = g(\tau), \qquad \tau^{**} - \delta \leqslant \tau \leqslant T.$$

So $h(\tau) \in C^\infty(0, T]$. $\square$

We conclude this paper by a remark.

Remark. We have proved the monotonicity and smoothness of the free boundary at some occasions. However, numerical experiments given by [7] show that the free boundary is monotone for all cases. A complete proof remains undiscovered.

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